5.6.2 Transformation Properties of Tensors

The last property of tensors that we need to consider is the manner in which they transform between different reference frames. This can be derived in a direct fashion that makes use of the fact that the tensors were defined in a way that is independent of the choice of reference frame, and hence tensors are geometrical objects (in the same way as four-vectors are) that have an existence in spacetime independent of any choice of reference frame. With that in mind, we can immediately write for the covariant components of a tensor T

$$\mathsf{T}(\vec{a}, \vec{b}, \vec{c}, \ldots) = T_{\mu\nu\alpha\ldots} a^{\mu} b^{\nu} c^{\alpha} \ldots = T_{\mu'\nu'\alpha'\ldots} a^{\mu'} b^{\nu'} c^{\alpha'} \ldots . \tag{5.99}$$

By using $a^{\mu'} = \Lambda^{\mu'}_{\nu} a^{\nu}$ and similarly for the other vector components, this becomes

$$T_{\mu\nu\alpha\ldots} a^{\mu} b^{\nu} c^{\alpha} \ldots = T_{\mu'\nu'\alpha'\ldots} \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} \Lambda^{\alpha'}_{\alpha} \ldots a^{\mu} b^{\nu} c^{\alpha} \ldots \tag{5.100}$$

As the vectors $\vec{a}, \vec{b}, \vec{c} \ldots$ are arbitrary, we have

$$T_{\mu\nu\alpha\ldots} = \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} \Lambda^{\alpha'}_{\alpha} \ldots T_{\mu'\nu'\alpha'\ldots} \tag{5.101}$$

In other words, the transformation is carried out in the same fashion as we have seen for the single index case (i.e. for the components of vectors). In a similar way (by use of $g^{\mu\nu}$ to raise indices), we can show for the contravariant components that

$$T^{\mu\nu\alpha\ldots} = \Lambda^{\mu}_{\mu'} \Lambda^{\nu}_{\nu'} \Lambda^{\alpha}_{\alpha'} \ldots T^{\mu'\nu'\alpha'\ldots} \tag{5.102}$$

The results Eq. (5.101) and Eq. (5.102), and a corresponding result for mixed components of the tensor T can be used as a test to see whether or not a multi-indexed quantity is, in fact, a tensor. We shall see how this can be implemented in the case of the Faraday tensor used to describe the electromagnetic field.